The proof follows from Lemmas 1 and 2. Given the constraint u_0, we obtain from inequality (12) the value $\mu_0 = u_0 c_0^{1/2} C_1^{-1} | b |^{-1}$, which guarantees the required control bound for any $t \in [0, T]$, $T \geqslant 1$. Then, using the bound (10), we compute R_0 and the time T that ensures the condition $| x_0 | \leqslant R_0 (\mu_0, T)$. To this end, it suffices to take $T \geqslant | x_0 |^2 \cdot \mu_0^{-2} c_0^{-1} c_1^2 + 1$. Substituting the corresponding value of μ_0, we finally obtain $T \geqslant | x_0 |^2 u_0^{-2} c_0^{-2} c_1^4 | b |^2 + 1$.

Example. Consider a point that moves with a bounded velocity along the horizontal directrix. Assume that the velocity of the point may change instantaneously within given bounds. There are m pendulums of various lengths attached to the point. Controlling the velocity of the point, it is required to move the system to the origin so that all the oscillations are damped. It can be verified that this system is stable and completely controllable (the controllability is proved in /5/). Therefore, for any bounded region in phase space, we can construct by our theorem a linear velocity controller which takes the system from any initial position in this region to the origin. The control on any of the realized trajectories will not exceed the specified value.

Remarks. 1°. If system (1) is unstable, then a control bounded by a given constant that takes a phase point to the origin does not necessarily exist. As an example, consider the equation $x^{\cdot} = x + u$, $| u | \leqslant u_0$, $x \in R^1$. For $| x_0 | > u_0$, the required control obviously does not exist. The conditions of the theorem are therefore very close to necessary.

2°. All the bounds and conclusions remain valid in the case when u is a vector and b is an appropriately dimensioned matrix.

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A SPECIAL CASE OF HYDRODYNAMIC STABILITY*

A.G. BERSHADSKII

The following dependence of the amplitude of the velocity perturbations on the supercriticality parameter: $A \sim \delta^{1/2}$ is typical of the case of the selfexcited oscillations which are generated when there is instability in the stationary flows of a viscous incompressible fluid. There is, however, a special case (it is investigated in this note) when this dependence is linear (as in the case of bifurcations in a stationary regime /1/). A condition is obtained for the existence of such selfexcited oscillations together with an algorithm which enables one to find their frequency and amplitude. In the case of these selfexcited oscillations there is a further difference from conventional hydrodynamic selfexcited oscillations in that sub- and supercritical regimes coexist in them and, at the same time, the subcritical selfexcited oscillations turn out to be unstable while the supercritical selfexcited oscillations are stable.

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The problem of the stability of a stationary flow of a viscous incompressible fluid can be reduced to an investigation of the stability of the zeroth solution of the equation /1, 2/

$$du/dt = Lu + F(u, u) \tag{1}$$

where u is an element of a certain Hilbert space, L is a linear operator in this space, and F is a bilinear operator in this space. If the spectrum of the operator L emerges into the right half plane then the primary stationary flow becomes unstable /2/. The operator L is selfadjoint. The eigenvectors and associated vectors of this operator form a complete system of vectors in the Hilbert space under consideration /2/. Let us expand the solution of Eq.(1) in a series in the above-mentioned system of vectors

$$u(t) = \sum_{k=1}^{\infty} a_k(t)\psi_k \tag{2}$$

It follows from the real nature of the physical fields that the complex eigenvalues ν always coexist with the complex conjugate vaues $\bar{\nu}$ in the spectrum of the operator L /2/: $L\psi = \lambda\psi$, $L\bar{\psi} = \bar{\nu}\bar{\psi}$. If the stationary flow admits of symmetry groups it is possible to arrange more than one pair of eigenvalues of the operator L on a straight line parallel to the imaginary axis /1/. Let a straight line exist close to the imaginary axis on which two such pairs are located: $\nu_1 = \delta + i\omega_0$; $\nu_2 = \delta + 2i\omega_0$, $\bar{\nu}_1$, $\bar{\nu}_2$, where the magnitude of $|\delta|$ is small. The remaining eigenvalues of the operator L lie to the left of this straight line and the imaginary axis. From what follows it will be clear that this satisfaction of the above-mentioned conditions with an accuracy $O(\delta^3)$ is sufficient for selfexcited oscillations to exist.

Let ν_1 and ν_2 be simple eigenvalues /2/. Then, by substituting the series (2) into (1) and by scalar multiplication by the eigenvectors of the operator L^+, which is adjoint to L, we get the equations for a_1 and a_2

$$\frac{da_j}{dt} = \nu_j a_j + \sum_{k,m=1}^{\infty} b_{jkm} a_k a_m, \quad j = 1, 2$$

$$d_{jkm} = (\varphi_j, F(\psi_k, \psi_m))/(\varphi_j, \psi_m); \quad L^+\varphi_j = \lambda_j \varphi_j$$

In deriving these equations, account has been taken of the fact that the function φ_l is orthogonal to all ψ_k, apart from ψ_1, and that φ_2 is orthogonal to all ψ_k, apart from ψ_2. The equations for $a_3 = \bar{a}_1$ and for $a_4 = \bar{a}_2$ are obtained by taking the complex conjugates. It can be shown, that, when the remaining parameters are fixed, the quantity $\delta \sim (\mathrm{Re} - \mathrm{Re}_c)$/1, 2/. We shall therefore refer to δ as the supercriticality parameter.

Let us make the change of variables $\Theta = \omega t$, where ω is the required frequency of the selfexciting oscillations and expand $a_n(\Theta)$ and ω in series in powers of δ (see /1, 2/ regarding the basis of such analytical expansions for problems of gas-dynamic stability)

$$a_n = \delta a_n^{(1)} + \delta^2 a_n^{(2)} + \ldots, \ \omega = \omega_0 + \delta\omega_1 + \delta^2\omega_2 + \ldots, \ b_{jkm}^{(0)} = b_{jkm}(\delta = 0)$$

By substituting these expansions into the equations, we get to the first order,

$$a_j^{(1)}(\Theta) = A_j \exp(ij\Theta) \tag{3}$$

The amplitudes A_j are found from the following orders of the expansion. It can be shown that, when $h > 4$,

$$a_n^{(1)}(\Theta) = (c_0 + c_1\Theta + \ldots + c_k\Theta^k)\exp\left(\lambda_n \frac{\Theta}{\omega_0}\right)$$

(the occurrence of a factor which is a polynomial in Θ is associated with the joining of the vectors /3/). Since $\mathrm{Re}\,\nu_n < 0$ when $n > 4$ then, when $n > 4$, the quantities $a_n^{(1)}(\Theta)$ decay with time.

The equations will subsequently be written out without terms which decay with time, since the latter are of no interest. The, to the second order, we get (differentiation with respect to Θ is denoted by a dot)

$$\dot{a}_j^{(2)} - ija_j^{(2)} = -\frac{\omega_1}{\omega_0}\dot{a}_j^{(1)} + \frac{\beta}{\omega_0} a_j^{(1)} + \sum_{n,m=1}^{4} b_{jnm}^{(0)} a_n a_m$$

where $\beta = 1$ for the supercritical case and $\beta = -1$ for the subcritical case. If one now substitutes the $a_j^{(1)}$ from (3) into the right-hand sides of these equations, the terms $\sim \exp(i\Theta)$ when $j = 1$ and the terms $\sim \exp(i2\Theta)$ when $j = 2$ will lead to resonances. The conditions for freedom from resonances

$$A_2 = b_{211}^{(0)} A_1^2/(2i\omega_1 - \beta) \tag{4}$$
$$(2i\omega_1 - \beta)(\beta - i\omega_1) + |A_1|^2 k = 0$$
$$k = k_1 + ik_2 = \operatorname{Re}\{(b_{123}^{(0)} + b_{132}^{(0)}) b_{211}^{(0)}\} + i \operatorname{Im}\{b_{123}^{(0)} + b_{132}^{(0)}\}$$

yield

$$|A_1|^2 = -\frac{3\beta}{k_2}\omega_1, \quad \omega_1 = \beta\left(\frac{3}{4}\frac{k_1}{k_2} \pm \sqrt{\frac{9}{16}\frac{k_1^2}{k_2^2} + \frac{1}{2}}\right) \tag{5}$$

The fact that it is not the quantity A_1 which has been found but its modulus is not important since the initial phase of one of the selfexciting oscillations of a doublet can be arbitrarily chosen and, in particular, A_1 may be taken as being real. It follows from the first relationship in (5) that its right-hand side must be positive. This is attainable for any sign of β by a choice of a suitable sign in the second relationship (5). Hence, sub- and supercritical selfexciting oscillations coexist in this case (unlike the usual self-exciting oscillations where sub- and supercritical selfexciting oscillations preclude one another). The question of the stability of the sub- and supercritical regimes is therefore particularly important here. In answering this question, the discussion may be confined to the first orders with respect to δ. Let us consider the equations

$$(\omega_0 + \delta\omega_1) a_j^{\cdot} = (\beta\delta + ij\omega_0) a_j + \delta \sum_{k,m=1}^{4} b_{jkm} (a_k^{(1)} a_m + a_m^{(1)} a_k)$$

which have been linearized on the investigated selfexciting oscillation, where $a_k^{(1)}(\Theta)$ are the modes of the investigated selfexciting oscillations in the first order with respect to δ. The Floquet representation for the evolution operator of these equations is /4, 5/:

$$V(\Theta) = Q(\Theta) \exp(G\Theta)$$

The stability or instability of the selfexciting oscillation is determined by the location of the spectrum of the operator G. the smallness of δ can be used to determine when the spectrum of the operator G gets into the right half-plane. Let us expand the operator in powers of δ:

$$G = G_0 + \delta G_1 + \ldots, \quad G_0 = i \operatorname{diag}\{1, 2\}$$

The eigenvalues of the operator G_0 are as follows: $\sigma_1^{(0)} = i$, $\sigma_2^{(0)} = 2i$ while the eigen-vectors corresponding to them, are $x_1 = (1, 0)$, $x_2 = (0, 1)$. The formula due to Krein /5/

$$G_1 = \frac{1}{4\pi^2} \int_0^{T_0} \oint_{\gamma_0} \oint_{\gamma_0} \frac{(\lambda - \mu) e^{(\mu-\lambda)\tau}}{1 - e^{(\mu-\lambda)\tau}} (G_0 - \lambda I)^{-1} A_1 (G_0 - \mu I)^{-1} \, d\tau \, d\lambda \, d\mu \tag{6}$$
$$T_0 = 2\pi\omega_0^{-1}, \quad A_1 = \omega_0^{-1} \operatorname{diag}\{\beta - i\omega_1, \ \beta - 2i\omega_1\}]$$

can be used to determining the operator G_1, where γ_0 is a contour which encompasses the spectrum of the operator G_0 fairly tightly.

It is obvious that harmonic perturbations of the right-hand side of the equation do not make any contributions to $\operatorname{Re}\{\sigma\} \sim \delta$.

If the spectrum of the operator G is expanded in a series in powers of $\delta: \sigma = \sigma^{(0)} + \delta\sigma^{(1)} + \ldots$, we obtain

$$\sigma^{(1)} = (y, G_1 x) \tag{7}$$

where x is an eigenvector of the operator G_0, y is an eigenvector of the operator which is adjoint to it, and the brackets here denote a two-dimensional scalar product. Then, by substituting the value of G_1 from (6) and (7) and evaluating the integrals, we obtain

$$\operatorname{Re}\{\sigma\} = -\beta\omega_0^{-1}\delta + \ldots.$$

It follows from this that, for small δ, the subcritical selfexcited oscillations $(\beta = -1)$ are unstable $(\operatorname{Re}\{\sigma\} > 0)$ while the supercritical selfexcited oscillations $(\beta = 1)$ are stable $(\operatorname{Re}\{\sigma\} < 0)$.

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THE FORMATION OF ZERO FREQUENCY INTERNAL WAVES DURING FREE CONVECTION IN A TEMPERATURE-STRATIFIED LIQUID*

A.V. KISTOVICH and YU.D. CHASHECHKIN

It is shown, as a result of an analysis of the equations of free convection in a temperature-stratified medium (TSM), that internal waves of zero frequency are formed when a thermal source is included together with a floating flare. The wavelength of these waves is calculated and the parameters of the transition processes are determined.

Zero-frequency internal waves, which are observed experimentally /1/, are an important element of convective flows which are generated by thermal sources in liquids with saline stratification. Only the parameters of a flare which floats above a localized source of heat have been calculated in a TSM /2/. There is interest in the possibility of the existence of zero-frequency internal waves which are excited by the thermal source in a TSM and in determining their parameters.

1. Formulation of the problem. The linearized system of convection equations in the TSM in a cylindrical system of coordinates at the origin of which a thermal source with a power P is located and where the gravitational vector $\mathbf{g}$ is directed opposite to the z-axis has the form

$$\partial\mathbf{u}/\partial t = -\frac{1}{\rho_0}\nabla p + \nu\Delta\mathbf{u} + \frac{\nu}{3}\nabla(\nabla\cdot\mathbf{u}) - \alpha T'\mathbf{g} \tag{1.1}$$

$$\partial T'/\partial t + \nabla\cdot(\mathbf{u}T_0(z)) = \chi\Delta T' + \frac{P}{c_p\rho_0}\frac{\delta(z)\,\delta(r)}{2\pi r}\theta(t)$$

$$\partial\rho/\partial t - \alpha\rho_0\nabla\cdot(\mathbf{u}T_0(z)) + \rho_0\nabla\cdot\mathbf{u} = -\frac{\alpha P}{c_p}\frac{\delta(z)\,\delta(r)}{2\pi r}\theta(t)$$

$$\rho = \rho_0(1-\alpha T),\quad T = T_0(z) + T',\quad T_0(z) = T_0(1 + z/(\alpha T_0\Lambda))$$

Here $\mathbf{u}$ is the velocity of the medium, p is the pressure after subtracting the hydrostatic pressure, $T, T_0(z)$ and T' are the total, stratifying and excess temperatures, T_0 and ρ_0 are the temperature and density of the medium at the level $z=0$, ρ is the density of the medium, α, χ and ν are the coefficients of thermal expansion, the thermal diffusivity and the kinematic viscosity, c_p is the heat capacity of the medium at constant pressure and Λ is the temperature stratification scale. The initial and boundary conditions, taken at infinity and the conditions on the functions $\mathbf{u}, p$ and T' are homogeneous.

The velocity field, which is axially symmetric can be represented in the form

$$\mathbf{u} = \mathbf{v} + \mathbf{w},\quad \mathbf{v} = -\nabla h,\quad w_r = -\partial\Phi/\partial r,\quad w_z = -\partial\Psi/\partial z$$

$$\Delta_r\Phi + \frac{\partial^2\psi}{\partial z^2} = 0,\quad \Delta_r = r^{-1}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

Here, w_r and w_z are the radial and vertical components of the solenoidal part of the

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